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ON THE PROBLEM OF VIBRATIONS OF A SLIGHTLY CAMBERED PLATE

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The influence of slight camber of the middle line of a transverse section on the natural vibrations frequency and mode of an infinitely long plate, clamped at the endfaces, which is vibrating under plane strain conditions, is examined on the basis of perturbation theory [1–4] in the special case when some frequency of vibration of the uncambered plate is double. The initial system is degenerate [2], a "small imperfection can cause a large effect" for it ([1], Vol. 1, Sect. 149). The problem under consideration is a particular case of the problem of the influence of a small change in shape on the vibrations of a shell having multiple natural frequencies.

A supplement to an assertion of the author of [5] on the separation of natural shell vibrations into quasi-transverse and quasi-tangential is also contained herein.

1. To determine the mode and frequencies in the case under consideration, we have from the general equations of shell vibrations [5]

$$(A^{(0)} + \psi A^{(1)} + \psi^2 A^{(2)})\,(v,\ w) = \lambda\,(v,\ w) \tag{1.1}$$

$$A^{(\nu)} = \| a_{ij}^{(\nu)} \| \qquad \nu = 0, 1, 2 \qquad i, j = 1,2$$

Let us present expressions for the nonzero elements of the matrix operators $A^{(\nu)}$

$$a_{11}^{(0)} = -\frac{d^2}{ds^2}, \qquad a_{22}^{(0)} = \frac{h_*^{\,2}}{3}\frac{d^4}{ds^4}, \qquad a_{12}^{(1)} = \frac{d(\varkappa\cdot)}{ds} - \frac{h_*^{\,2}}{3}\varkappa\frac{d^3}{ds^3}$$

$$a_{21}^{(1)} = \frac{h_*^2}{3}\frac{d^3(\varkappa\cdot)}{ds^3} - \varkappa\frac{d}{ds}, \qquad a_{11}^{(2)} = -\frac{h_*^2}{3}\varkappa\frac{d^2(\varkappa\cdot)}{ds^2}, \qquad a_{22}^{(2)} = \varkappa^2 \qquad \text{(cont.)}$$

$$\varkappa(s) = \frac{\varkappa_0(s)}{\psi}, \qquad \psi = \left(\int_{-1/2}^{1/2} \varkappa_0^2\, ds\right)^{1/2}$$

$$\varkappa_0(s) = k_0 / s, \qquad s = s_0 / l, \qquad h_* = h / l$$

Here v, w are the tangential and normal displacements to the middle line of the plate cross section, s_0 is the arclength (Fig. 1) along the middle line, h is half the plate thickness, k_0 is the curvature of the middle line of the cross section.

We have the following expression for the vibration frequency ω :

$$\omega^2 = \lambda E / \rho l^2 (1 - \sigma^2)$$

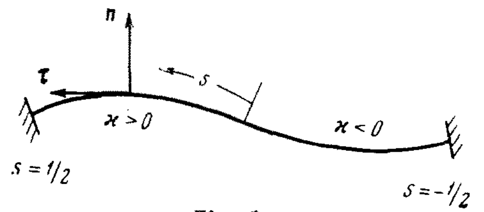

Fig. 1

Here ρ is the density of the plate material, σ is the Poisson's ratio, and E is the Young's modulus.

The boundary conditions are the following:

$$v(-{}^1/_2) = v({}^1/_2) = w(-{}^1/_2) = \qquad (1.2)$$
$$= w'(-{}^1/_2) = w({}^1/_2) = w'({}^1/_2) = 0$$

It is considered throughout that the vectors (v, w) belong to a Hilbert space of vector-functions with the scalar product ([6], pp. 19, 27)

$$((v_1, w_1)\cdot(v_2, w_2)) = \int_{-1/2}^{1/2} v_1 \bar{v}_2\, ds + \int_{-1/2}^{1/2} w_1 \bar{w}_2\, ds$$

and hence the operators $A^{(\nu)}$ (the boundary conditions (1.2)) are self-adjoint.

Let us introduce eigenvalues and normalized eigenvectors of the operator $A^{(0)}$ corresponding to the longitudinal vibrations of a rod

$$\lambda_{0m} = (m\pi)^2, \quad (v_{0m}, 0) \qquad (m = 1, 2, 3, \ldots)$$

and the transverse vibrations of a rod

$$\lambda_{0k} = {}^1/_3 h_*^2 \mu_k^4, \quad (0, w_{0k}) \quad (k = 1, 2, 3, \ldots)$$

As is known

$$\cos(\mu_k)\,\mathrm{ch}(\mu_k) = 1, \qquad \mu_k \approx (k + {}^1/_2)\pi$$

For some m and k let

$$\lambda_{0m} = \lambda_{0k} = \lambda_0 \tag{1.3}$$

A two-dimensional proper subspace L_{λ_0} of the operator $A^{(0)}$ corresponds to the eigenvalue λ_0; as the basis in this subspace we select the vectors

$$\chi_1 = (v_{0m}, 0), \qquad \chi_2 = (0, w_{0k})$$

The matrix

$$\begin{vmatrix} 0 & a \\ a & 0 \end{vmatrix} \qquad a = \int_{-1/2}^{1/2} v_{0m}(\varkappa w_{0k})'\, ds - \frac{h_*^2}{3}\int_{-1/2}^{1/2} \varkappa v_{0m} w_{0k}'''\, ds \tag{1.4}$$

corresponds to the operator $A^{(1)}$ in the subspace L_{λ_0}.

It has been assumed in the derivation of (1.4) that the function $\varkappa$ "does not spoil" the possibility of integration by parts in (1.4) with the terms outside the integral vanishing because of the boundary conditions (1.2) (analogous requirements are imposed on $\varkappa$ in verifying self-adjointness).

For small ψ we seek the solution (1.1) in the form

$$(v,\ w) = u_0 + \psi\,(v_1,\ w_1) + \psi^2\,(v_2,\ w_2) + \ldots$$
$$\lambda = \lambda_0 + \psi\lambda_1 + \psi^2\lambda_2 + \ldots,\quad u_0 = \alpha\chi_1 + \beta\chi_2 \qquad (1.5)$$

We henceforth limit ourselves throughout to seek vectors u_0 and the first nonzero corrections to λ_0 by considering the parameter ψ "sufficiently small". If $a \neq 0$, we have

$$\lambda_1 = \pm\, a, \qquad \alpha\,/\,\beta = \pm\, 1 \qquad (1.6)$$

Let $a = 0$. Seeking the solutions for which the representation (1.5) is valid as before, we set $v = \psi v^0$ (another possibility is to use the substitution $w = \psi w^0$), then we obtain instead of (1.1)

$$(B^{(0)} + \psi^2 B^{(1)})\,(v^0,\ w) = \lambda\,(v^0,\ w)$$

where

$$B^{(\nu)} = \|\, b_{ij}^{(\nu)} \|\quad \nu = 0,1 \quad i, j = 1, 2$$

The nonzero elements b_{ij} are

$$b_{11}^{(0)} = -\frac{d^2}{ds^2}, \quad b_{12}^{(0)} = \frac{d\,(\varkappa\,\cdot)}{ds} - \frac{h_*^{\,2}}{3}\,\varkappa\,\frac{d^3}{ds^3}, \quad b_{22}^{(0)} = \frac{h_*^{\,2}}{3}\,\frac{d^4}{ds^4}$$

$$b_{11}^{(1)} = -\frac{h_*^{\,2}}{3}\,\varkappa\,\frac{d^2\,(\varkappa\,\cdot)}{ds^2}, \quad b_{21}^{(1)} = \frac{h_*^{\,2}}{3}\,\frac{d^3\,(\varkappa\,\cdot)}{ds^3} - \varkappa\,\frac{d}{ds}, \quad b_{22}^{(1)} = \varkappa^2$$

If $a = 0$, then λ_0 is a double eigenvalue of the operator $B^{(0)}$, to which the orthonormalized eigenvectors g_1, g_2 correspond

$$g_1 = (v_{0m},\ 0),\ g_2 = \mu\,(p_0,\ w_{0k}),\ \mu = (1 + (p_0\cdot p))^{-1/2}$$

where p_0 is a solution of the equation

$$p_0'' + \lambda_0 p_0 = (\varkappa\, w_{0k})' - {}^1/_3\, h_*^{\,2}\,\varkappa w_{0k}''' \ (= p)$$

with the boundary conditions taken for v, and such that

$$(p_0\cdot v_{0m}) = 0$$

If $a = 0$, then λ_0 is also a double eigenvalue of the operator $(B^{(0)})^*$ conjugate to $B^{(0)}$ The orthonormalized eigenvectors of the operator $(B^{(0)})^*$ corresponding to λ_0, are:

$$e_1 = \gamma\,(v_{0m},\ r_0),\ e_2 = (0,\ w_{0k}),\ \gamma = (1 + (r_0\cdot r_0))^{-1/2}$$

Here r_0 is the solution of the equation

$${}^1/_3 h_*^{\,2}\, r_0'''' - \lambda_0 r_0 = \varkappa\, v_{0m}' - {}^1/_3 h_*^{\,2}\,(\varkappa\, v_{0m})''' \quad (= r)$$

with boundary conditions for w, which is orthogonal to w_{0k}.

Associated vectors for eigenvectors of the operator $B^{(0)}$ are missing. On the basis of results in [4] (pp. 61-62), we seek for small ψ

$$(v^0,\ w) = u^0 + \psi^2\,(v_1^{\,0},\ w_1^*) + \psi^4\,(v_2^{\,0},\ w_2^*) + \ldots$$
$$\lambda = \lambda_0 + \psi^2\lambda_2 + \psi^4\lambda_4 \ldots,\ u^0 = \alpha^0 g_1 + \beta^0 g_2$$

The quantities λ_2 are eigenvalues, and u_0 the corresponding eigenvectors of the operator C in a two-dimensional subspace with basis g_1, g_2 such that the matrix

$$C = \left\|\begin{matrix} \gamma^{-1}\,(B^{(1)}\,g_1\cdot e_1) & \gamma^{-1}\,(B^{(1)}\,g_2\cdot e_1) \\ 0 & \mu^{-1}\,(B^{(1)}\,g_2\cdot e_2) \end{matrix}\right\|$$

corresponds to it in this basis.

We have

$$\lambda_{21} = \gamma^{-1}\,(B^{(1)}\,g_1\cdot e_1) = \frac{h_*^{\,2}}{3}\int_{-1/2}^{1/2} [(\varkappa v_{0m})']^2\, ds - \int_{-1/2}^{1/2} r r_0\, ds$$

$$\lambda_{22} = \mu^{-1}(B^{(1)} g_2 \cdot e_2) = \int_{-1/2}^{1/2} (\varkappa w_{0k})^2 ds + \int_{-1/2}^{1/2} p p_0 ds \qquad \text{(cont.)}$$

If one of the quantities λ_2 turns out to be zero, then to obtain the appropriate first correction to the eigenvalue λ_0 further analyses are required. If $\lambda_{21} = \lambda_{22}$, the determination of u^0 from an analysis of the eigenvectors of the operator C is not always possible. Both these cases require futher analysis, and are not considered herein.

In all the other cases the eigenvectors corresponding to λ_{21}, λ_{22} are

$$(v, w)_1 = (v_{0m}, 0) + \psi(0, (\gamma\mu)^{-1}(\lambda_{21} - \lambda_{22})^{-1}(B^{(1)} g_2 \cdot e_1) w_{0k} + r_0) + \ldots$$
$$(v, w)_2 = (0, w_{0k}) + \psi((\gamma\mu)^{-1}(\lambda_{22} - \lambda_{21})^{-1}(B^{(1)} g_2 \cdot e_1) v_{0m} + p_0, 0) + \ldots$$

Let the panel curvature $1 / R$ be constant, then $\varkappa = 1$, $\psi = l / R$; if m, k are of different parity, and $k \gg m$, then

$$a \approx (-1)^{(m+k+1)/2} 4 \sqrt{2}\, m / k$$

The eigenvectors (v_{0m}, w_{0k}) are taken thus:

$$\sqrt{2}(\sin(m\pi s), \cos(\mu_k s)), \quad m = 2m', \quad m' = 1, 2, 3, \ldots$$
$$\sqrt{2}(\cos(m\pi s), \sin(\mu_k s), \quad m = 2m' - 1, \quad m' = 1, 2, 3, \ldots$$

Since $a \neq 0$, then (1.6) is valid. Now, if the parity of m, k is identical, then $a = 0$; limiting ourselves to the case of even m, k under the condition $k \gg m$, we obtain

$$\lambda_{21} \approx 1 + (m / k)^4, \quad \lambda_{22} \approx -3 (m / k)^2$$

the corresponding vibration modes go over for $\psi \to 0$ to pure transverse, and pure tangential.

Problems analogous to that considered above, for shells of sufficiently general form, which have multiple vibration frequencies, are also expediently solved on the basis of perturbation theory.

In those cases when the degeneration is complete, or due partially to symmetry, the simplifications induced by taking account of the symmetry of the unperturbed and perturbed problems should be used ([7], Sect. 20, 22 ; [8], Chap. 20).

There are certain analyses for systems whose degeneration is due to symmetry in [1] (Vol. 1, Sect. 209, 221); Rayleigh uses the principle of stationarity of the frequencies. The experiments in [9] can be illustrations of these analyses.

Let us note that in the case of a shell with multiple frequencies the vibration modes can have some peculiarities ([1], [10], Chap. 4, [11−15]).

2. It is asserted in [5] that the natural vibration modes of shells (under the specific conditions mentioned in [5]) are subdivided into quasi-transverse and quasi-tangential. The cases examined above, when the transverse and tangential vibrations are superposed in a 1:1 relationship, show that this is not always the case. Other analogous examples can be obtained as follows. Let us consider axisymmetric natural vibrations of a circular cylindrical shell. We have $(D^{(0)} + \sigma D^{(1)})(u, w) = q(u, w)$

$$D^{(\nu)} = \| d_{ij}^{(\nu)} \| \quad \nu = 0, 1 \quad i, j = 1, 2$$
$$d_{11}^{(0)} = -\frac{d^2}{dx^2}, \quad d_{22}^{(0)} = \frac{h_*^2}{3}\frac{d^4}{dx^4} + 1 \qquad (2.1)$$
$$d_{12}^{(1)} = -d_{21}^{(1)} = \frac{d}{dx}, \quad d_{12}^{(0)} = d_{21}^{(0)} = d_{11}^{(1)} = d_{22}^{(1)} = 0$$
$$x = s / R, \; h_* = h / R, \; 0 \leqslant s \leqslant l, \qquad q = \rho R^2 \omega^2 (1 - \sigma^2) / E$$

Here u, w are the displacements along and normal to the generator, s is the arclength along the generator, R the shell radius, h half its thickness, and the quantities ρ, ω, σ, E are the same as in Sect. 1. We use the notation

$$q_{0m}, q_{0k} \qquad (m,\ k = 1, 2, 3, \ldots)$$

to denote the simple eigenvalues of the operator $D^{(0)}$ corresponding to pure tangential and pure transverse vibrations, respectively. For some m, k let

$$q_{0m} = q_{0k} \tag{2.2}$$

then for sufficiently small σ the displacement vector, in a first approximation, is equal $(\alpha u_{0m},\ \beta w_{0k})$, where α, β are not simultaneously zero in the general case.

Let us examine specific examples.

1. A shell is clamped at the endfaces $(L = l \,/\, R)$

$$u\ (0) = u\ (L) = w\ (0) = w'\ (0) = w\ (L) = w'\ (L) = 0 \tag{2.3}$$

Condition (2. 2) yields

$$(m\pi \,/\, L)^2 = 1 + {}^1/_3 h_*^2\, (\mu_k \,/\, L)^4$$

If m, k are of different frequency, then $\alpha \,/\, \beta = \pm 1$. When the parity of m, k is the same, the analysis can be conducted analogously to the analysis for $a = 0$ in Sect. 1.

2. Shell supported at the endfaces

$$u'\ (0) = u'\ (L) = w\ (0) = w''\ (0) = w\ (L) = w''\ (L) = 0$$

Flügge [16] found the following solutions for this case:

$$u = \alpha \cos\, (n\pi x \,/\, L),\ w = \beta \sin\, (n\pi x \,/\, L) \qquad (n = 1, 2, 3, \ldots)$$

Condition (2. 2) yields

$$(m\pi \,/\, L)^2 = 1 + {}^1/_3 h_*^2\ (k\pi \,/\, L)^4$$

If $m \neq k$, then as $\sigma \to 0$ the limit values of the normalized eigenvectors are

$$\sqrt{2}\, (\cos\, (m\pi x \,/\, L),\ 0), \qquad \sqrt{2}\, (0,\ \sin\, (k\pi x \,/\, L) \tag{2.4}$$

Now, let $m = k$

$$(k\pi \,/\, L)^2 = 1 + {}^1/_3 h_*^2\, (k\pi \,/\, L)^4 \tag{2.5}$$

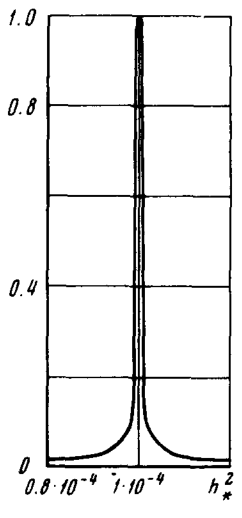

Fig. 2

In this case the exact solution can be written as

$$q = (k\pi \,/\, L)^{\,2}\, [1 \pm \sigma\, (L \,/\, k\pi)],\ \ \alpha \,/\, \beta = \pm 1$$

Because of the smallness of h_* we can select $L \approx k\pi$ such that (2. 5) will be satisfied, and the appropriate modes with the number of half-waves k will be a superposition of tangential and transverse vibrations in a "one to one" relationship. However, as L increases the shell "transforms" sufficiently rapidly into a thin-walled rod.

Equation (2. 5) also has the root

$$k \approx \sqrt{3}L\, (\pi h_*)^{-1}$$

The corresponding solutions of the system (2. 1) can no longer be considered as the solutions defining the frequencies and modes of shell vibrations. However, examining such solutions, an interesting regularity can be observed.

For some L, $h_*^{2(0)}$ and k (large), let (2. 5) be satisfied, then $\alpha/\beta = \pm 1$. However, with only a small change of h_*^2, the eigenvectors of the system (2. 1) become close to one of the vectors (2.4) almost immediately. Fig. 2 shows the change

$$\alpha / \beta \quad \text{for} \quad h_*^2 > h_*^{2(0)}, \quad \beta / \alpha \quad \text{for} \quad h_*^2 < h_*^{2(0)}$$

for one of the branches of the solution when h_*^2 is altered in the neighborhood of $h_*^{2(0)}$ and

$$(\pi / L) \approx 1, \ h_*^{2(0)} \approx 10^{-4}, \ k = 173$$

In connection with the character of the curve in Fig. 2, see [1] (Vol. 1, p. 90). If $k = 1$, the appropriate curves are almost horizontal. By varying L (for $h_*^2 = h_*^{2(0)}$) we obtain an analogous character of the dependence $\alpha/\beta = f(L)$ (if h_*^2 and L are changed simultaneously, the effect in Fig. 2 may not even be obtained).

In general, if the natural vibrations for some shell are subdivided, as is assumed in [5], into quasi-transverse and quasi-tangential, the possibility of "binding" these two kinds of vibrations together should not be lost sight of. Such a binding together can be due to the closeness between frequencies of some vibration modes of both kinds.

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